Some problems from Prelim 2 of Fall 2013

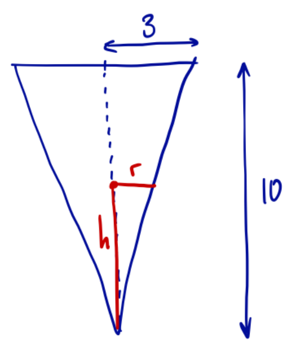
October 31, 2016

Problem 1. **(d)** We need to find derivative of $y = x^{2^x}$. Using, for example, logarithmic differentiation we get $\ln y = 2^x \ln x$, and so

$$\begin{aligned}(\ln y)' &= \frac{1}{y} \cdot y' \\ &= 2^x \cdot \ln 2 \ln x + 2^x \cdot \frac{1}{x} \\ &= 2^x(\ln 2 \ln x + \frac{1}{x})\end{aligned}$$

So, we have $\frac{1}{y} \cdot y' = 2^x(\ln 2 \ln x + \frac{1}{x})$, and therefore $y' = x^{2^x} \cdot 2^x(\ln 2 \ln x + \frac{1}{x})$.

Problem 2. **(a)** The formula for the area of a circle is $A = \pi r^2$. We need to find the rate of change $\frac{dA}{dt}$ after 2 hours. We have $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$. To find $\frac{dr}{dt}$ we can relate r to h, which will help since we are given that $\frac{dh}{dt} = 0.5$. Using similar triangles (see picture below) we get $\frac{3}{10} = \frac{r}{h}$, i.e. $r = \frac{3}{10}h$.



Therefore, $\frac{dr}{dt} = \frac{3}{10}\frac{dh}{dt} = \frac{3}{20}$. Finally, after 2 hours the height will be $h = 2 \cdot 0.5 = 1m$, and so r will be $r = \frac{3}{10} \cdot 1 = \frac{3}{10}$. Putting it all together, we get

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi \cdot \frac{3}{10} \cdot \frac{3}{20} = \frac{9\pi}{100}$$

(b) Now we are interested in computing the rate of change of the volume $V = \frac{1}{3}\pi r^2 h$ at the moment when the hole fills up, i.e. at the moment when $h = 10$ and $r = 3$. Differentiating and using the information we have found in part (a), we get

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3} r^2 \frac{dh}{dt} = \frac{2}{3}\pi \cdot 3 \cdot 10 \cdot \frac{3}{20} + \frac{1}{3}\pi \cdot 3^2 \cdot \frac{1}{2} = \frac{9}{2}\pi$$

Problem 4. (c) We need to find the derivative of $p(x) = \sqrt{f^{-1}(x) + 1}$ at $x = 2$ using the information in the following table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-2	-6	$1/2$	10	-7
-1	-2	0	3	1
0	2	-1	$1/3$	4
1	$5/2$	-4	8	2
2	11	3	2	5

Using the chain rule,

$$\frac{dp}{dx} = \frac{1}{2}\frac{1}{\sqrt{f^{-1}(x) + 1}} \cdot \frac{d}{dx}(f^{-1}(x) + 1) = \frac{1}{2}\frac{1}{\sqrt{f^{-1}(x) + 1}} \cdot \frac{1}{f'(f^{-1}(x))}.$$

Therefore, at $x = 2$ we will have

$$\frac{dp}{dx}(2) = \frac{1}{2}\frac{1}{\sqrt{f^{-1}(2) + 1}} \cdot \frac{1}{f'(f^{-1}(2))}$$

Since we are given $f(0) = 2$ we know that $f^{-1}(2) = 0$. As a result,

$$\frac{dp}{dx}(2) = \frac{1}{2}\frac{1}{\sqrt{0 + 1}} \cdot \frac{1}{f'(0)} = \frac{3}{2}$$